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Superluminal motion in the quasar 3C273

"The cowboys have a way of trussing up a steer or a pugnacious bronco which fixes the brute so that it can neither move nor think. This is the hog-tie, and it is what Euclid did to geometry."
Eric Temple Bell (1883–1960), *The Search For Truth* (1934).

General relativity endows spacetime with a causal structure described by observer-invariant *light cones.* This locally incorporates the theory of special relativity: the velocity of light is the same for all observers. Points *inside* a light cone are causally connected with its vertex, while points *outside* the same light cone are out-of-causal contact with its vertex. Light describes null-generators *on* the light cone. This simple structure suffices to capture the kinematic features of special relativity. We illustrate these ideas by looking at relativistic motion in the nearby quasar 3C273.

1.1 Lorentz transformations

Maxwell's equations describe the propagation of light in the form of electromagnetic waves. These equations are linear. The Michelson–Morley experiment[372] shows that the velocity of light is constant, independent of the state of the observer. Lorentz derived the commensurate linear transformation on the coordinates, which leaves Maxwell equations form-invariant. It will be appreciated that form invariance of Maxwell's equations implies invariance of the velocity of electromagnetic waves. This transformation was subsequently rederived by Einstein, based on the stipulation that the velocity of light is the same for any observer. It is non-Newtonian, in that it simultaneously transforms all four spacetime coordinates.

The results can be expressed geometrically, by introducing the notion of light cones. Suppose we have a beacon that produces a single flash of light in all directions. This flash creates an expanding shell. We can picture this in a spacetime

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diagram by plotting the cross-section of this shell with the x-axis as a function of time – two diagonal and straight lines in an inertial setting (neglecting gravitational effects or accelerations). The interior of the light cone corresponds to points interior to the shell. These points can be associated with the centre of the shell by particles moving slower than the speed of light. The interior of the light cone is hereby causally connected to its vertex. The exterior of the shell is out-of-causal contact with the vertex of the light cone. This causal structure is local to the vertex of each light cone, illustrated in Figure (1.1).

Light-cones give a geometrical description of causal structure which is observer-invariant by invariance of the velocity of light, commonly referred to as "covariance". Covariance of a light cone gives rise to a linear transformation of the spacetime coordinates of two observers, one with a coordinate frame $K(t,x)$ and the other with a coordinate frame $K'(t',x')$. We may insist on coincidence of K and K' at $t = t' = 0$, and use geometrical units in which $c = 1$, whereby

$$\mathrm{sign}(x^2 - t^2) = \mathrm{sign}(x'^2 - t'^2). \tag{1.1}$$

The negative (positive) sign in (1.1) corresponds to the interior (exterior) of the light cone. The light cone itself satisfies

$$x^2 - t^2 = x'^2 - t'^2 = 0. \tag{1.2}$$

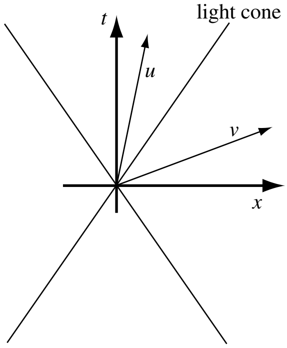

Figure 1.1 The local causal structure of spacetime is described by a light cone. Shown are the future and the past light cone about its vertex at the origin of a coordinate system (t, x). Vectors u within the light cone are timelike ($x^2 - t^2 < 0$); vectors v outside the light cone are spacelike ($x^2 - t^2 > 0$). By invariance of the velocity of light, this structure is the same for all observers. The linear transformation which leaves the signed distance $s^2 = x^2 - t^2$ invariant is the Lorentz transformation – a four-dimensional transformation of the coordinates of the frame of an observer.

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A linear transformation between the coordinate frames of two observers which preserves the local causal structure obtains through Einstein's invariant distance

$$s^2 = -x^2 + t^2. \tag{1.3}$$

This generalizes Eqns (1.1) and (1.2). Remarkably, this simple ansatz recovers the Lorentz transformation, derived earlier by Lorentz on the basis of invariance of Maxwell's equations. The transformation in the invariant

$$x^2 - t^2 = x'^2 - t'^2 \tag{1.4}$$

can be inferred from rotations, describing the invariant $x^2 + y^2 = x'^2 + y'^2$ in the (x, y)-plane, as the hyperbolic variant

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \tag{1.5}$$

The coordinates $(t, 0)$ in the observer's frame K correspond to the coordinates (t', x') in the frame K', such that

$$-\frac{x'}{t'} = \tanh\lambda. \tag{1.6}$$

This corresponds to a velocity $v = \tanh\lambda$ in terms of the "rapidity" λ of K' as seen in K. The matrix transformation (1.4) can now be expressed in terms of the relative velocity v,

$$t' = \Gamma(t - vx), \quad x' = \Gamma(x - vt), \tag{1.7}$$

where

$$\Gamma = \frac{1}{\sqrt{1 - v^2}} \tag{1.8}$$

denotes the Lorentz factor of the observer with three-velicity v.

The trajectory in spacetime traced out by an observer is called a world-line, e.g. that of K along the t-axis or the same observer as seen in K' following (1.8). The above shows that the tangents to world-lines – four-vectors – are connected by Lorentz transformations. The Lorentz transformation also shows that $v = 1$ is the limiting value for the relative velocity between observers, corresponding to a Lorentz factor Γ approaching infinity.

Minkowski introduced the world-line $x^b(\tau)$ of a particle and its tangent according to the velocity four-vector

$$u^b = \frac{dx^b}{d\tau}. \tag{1.9}$$

Here, we use a normalization in which τ denotes the eigentime,

$$u^2 = -1. \tag{1.10}$$

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At this point, note the Einstein summation rule for repeated indices:

$$u^b u_b = \Sigma_{b=0}^3 u^b u_b = \eta_{ab} u^a u^b \tag{1.11}$$

in the Minkowski metric

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.12}$$

The Minkowski metric extends the Euclidian metric of a Cartesian coordinate system to four-dimensional spacetime. By (1.10) we insist

$$(u^x)^2 + (u^y)^2 + (u^z)^2 - (u^t)^2 = -1, \tag{1.13}$$

where $u^b = (u^t, u^x, u^y, u^z)$. In one-dimensional motion, it is often convenient to use the hyperbolic representation

$$u^b = (u^t, u^x, 0, 0) = (\cosh\lambda, \sinh\lambda, 0, 0) \tag{1.14}$$

in terms of λ, whereby the particle obtains a Lorentz factor $\Gamma = \cosh\lambda$ and a three-velocity

$$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{u^x}{u^t} = \tanh\lambda. \tag{1.15}$$

The Minkowski velocity four-vector u^b hereby transforms according to a Lorentz transformation ($d\tau$ is an invariant in (1.9)). We say that u^b is a covariant vector, and that the normalization $u^2 = -1$ is a Lorentz invariant, also known as a scalar.

To summarize, Einstein concluded on the basis of Maxwell's equations that spacetime exhibits an invariant causal structure in the form of an observer-invariant light cone at each point of spacetime. Points inside the light cone are causally connected to its vertex, and points outside are out-of-causal contact with its vertex. This structure is described by the Minkowski line-element

$$s^2 = x^2 + y^2 + z^2 - t^2, \tag{1.16}$$

which introduces a Lorentz-invariant signed distance in four-dimensional spacetime (t, x, y, z) following (1.12). In attributing the causal structure as a property intrinsic to spacetime, Einstein proposed that *all* physical laws and physical observables are observer-independent, i.e. obey invariance under Lorentz transformations. This invariance is the principle of his theory of special relativity. Galileo's picture of spacetime corresponds to the limit of slow motion or, equivalently, the singular limit in which the velocity of light approaches infinity – back to Euclidean geometry and Newton's picture of spacetime.

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1.2 Kinematic effects

In Minkowski spacetime, rapidly moving objects give rise to apparent kinematic effects, representing the intersections of their world-lines with surfaces Σ_t of constant time in the laboratory frame K. In a two-dimensional spacetime diagram (x, t), Σ_t corresponds to horizontal lines parallel to the x-axis.

Consider an object moving uniformly with Lorentz factor Γ as shown in Figure (1.2) such that its world-line – a straight line – intersects the origin. The *lapse* in eigentime τ in the motion of the object from Σ_0 to Σ_t is given by

$$\tau = \int_0^t \frac{ds}{dt}\, dt = \sqrt{-(t, vt)^2} = t\sqrt{1-v^2}, \tag{1.17}$$

or

$$\frac{\tau}{t} = \frac{1}{\Gamma}. \tag{1.18}$$

Moving objects have a smaller lapse in eigentime between two surfaces of constant time, relative to the static observer in the laboratory frame. Rapidly moving elementary particles hereby appear with enhanced decay times. This effect is known as *time-dilation*.

The distance between two objects moving uniformly likewise depends on their common Lorentz factor Γ as seen in the laboratory frame K, as shown in

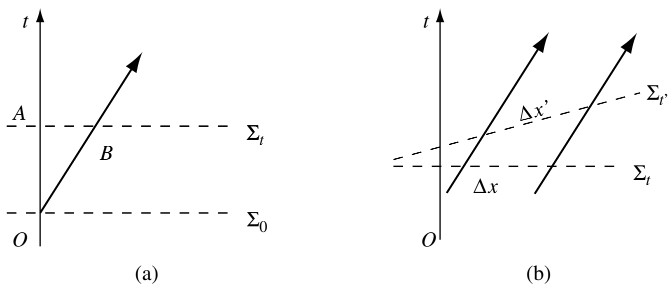

Figure 1.2 (*a*) Time dilation is described by the lapse in eigentime of a moving particle (arrow) between two surfaces of constant time Σ_0 and Σ_t in the laboratory frame K. The distance beteen these to surfaces in K is t, corresponding to O and A. The lapse in eigentime is t/Γ upon intersecting Σ_0 at O and Σ_t at B, where Γ is the Lorentz factor of the particle. Moving clocks hereby run slower. (*b*) The distance between two parallel world-lines (arrows) is the distance between their points of intersection with surfaces of constant time: Σ_t in K and $\Sigma_{t'}$ in the comoving frame K'. According to the Lorentz transformation, $\Delta x = \Delta x'/\Gamma$, showing that moving objects appear shortened and, in the ultrarelativistic case, become so-called "pancakes."

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Figure (1.2). According to (1.7), the distance Δx between them as seen in K is related to the distance $\Delta x'$ as seen in the comoving frame K' by

$$\Delta x = \Delta x'/\Gamma. \tag{1.19}$$

Hence, the distance between two objects in uniform motion appears reduced as seen in the laboratory frame. This effect is known as the *Lorentz contraction*. Quite generally, an extended blob moving relativistically becomes a "pancake" as seen in the laboratory frame.

1.3 Quasar redshifts

Quasars are highly luminous and show powerful one-sided jets. They are now known to represent the luminous center of some of the active galaxies. These centers are believed to harbor supermassive black holes.

The archetype quasar is 3C273 at a redshift of $z = 0.158$. The redshift is defined as the relative increase in the wavelength of a photon coming from the source, as seen in the observer's frame: if λ_0 denotes the rest wavelength in the frame of the quasar, and λ denotes the wavelength in the observer's frame, we may write

$$1 + z = \frac{\lambda}{\lambda_0}. \tag{1.20}$$

The quasar 3C273 shows a relative increase in wavelength by about 16%. This feature is achromatic: it applies to any wavelength.

We can calculate z in terms of the three-velocity v with which the quasar is receding away from us. Consider a single period of the photon, as it travels a distance λ_0 in the rest frame. The null-displacement (λ_0, λ_0) *on* the light cone (in geometrical units) corresponds by a Lorentz transformation to

$$\Gamma(\lambda_0 + v\lambda_0), \quad \Gamma(\lambda_0 + v\lambda_0). \tag{1.21}$$

Note the plus sign in front of $v\lambda$ for a receding velocity of the quasar relative to the observer. The observer measures a wavelength

$$\lambda = \lambda_0 \Gamma(1 + v) = \lambda_0 \sqrt{\frac{1+v}{1-v}}. \tag{1.22}$$

It is instructive also to calculate the redshift factor z in terms of a redshift in energy. Let p_a denote the four-momentum of the photon, which satisfies $p^2 = 0$ as it moves along a null-trajectory on the light cone. Let also u^a and v^a denote the velocity four-vectors of of the quasar and that of the observer, respectively. The energies of the photon satisfy

$$\epsilon_0 = -p_a u^a, \quad \epsilon = -p_a v^a. \tag{1.23}$$

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The velocity four-vectors u^a and v^a are related by a Lorentz transformation

$$v^a = \Lambda_a^b u^b, \quad \tanh\lambda = -v \tag{1.24}$$

in the notation of (1.5). It follows that

$$\epsilon = -p_a \Lambda_c^a u^c = -\eta_{ab} p^a \Lambda_c^a u^c. \tag{1.25}$$

This is a *scalar* expression, in view of complete contractions over all indices. We can evaluate it in any preferred frame. Doing so in the frame of the quasar, we have $p^a = \epsilon_0(1,1)$ and $u^b = (1,0)$. Hence, the energy in the observer's frame satisfies

$$\epsilon = \epsilon_0(\cosh\lambda - \sinh\lambda) = \epsilon_0\sqrt{\frac{1-v}{1+v}}. \tag{1.26}$$

Together, (1.22) and (1.26) obey the relationship $\epsilon = 2\pi/\lambda$, where $\epsilon_0\lambda_0 = 2\pi$.

1.4 Superluminal motion in 3C273

The quasar 3C273 is a variable source. It ejected a powerful synchrotron emitting blob of plasma in 1977, shown in Figure (1.3)[412]. In subsequent years, the angular displacement of this blob was monitored. Given the distance to 3C273 (based on cosmological expansion, as described by the Hubble constant), the velocity projected on the sky was found to be

$$v_\perp = (9.6 \pm 0.8) \times c. \tag{1.27}$$

An elegant geometrical explanation is in terms of a relativistically moving blob, moving close to the line-of-sight towards the observer, given by R. D. Blandford, C. F. McKee and M. J. Rees[65]. Consider two photons emitted from the blob moving towards the observer at consecutive times. Because the second photon is emitted while the blob has moved closer to the observer, it requires less travel time to reach the observer compared with the preceding photon. This gives the blob the appearance of rapid motion. We can calculate this as follows, upon neglecting the relative motion between the observer and the quasar. (The relativistic motion of the ejecta is much faster than that of the quasar itself.)

Consider the time-interval Δt_e between the emission of the two photons. The associated time-interval Δt_r between the times of receiving these two photons is reduced by the distance $D_{||} = v\cos\theta\Delta t_e$ along the line-of-sight traveled by the blob:

$$\Delta t_r = \Delta t_e - v\cos\theta\Delta t_e, \tag{1.28}$$

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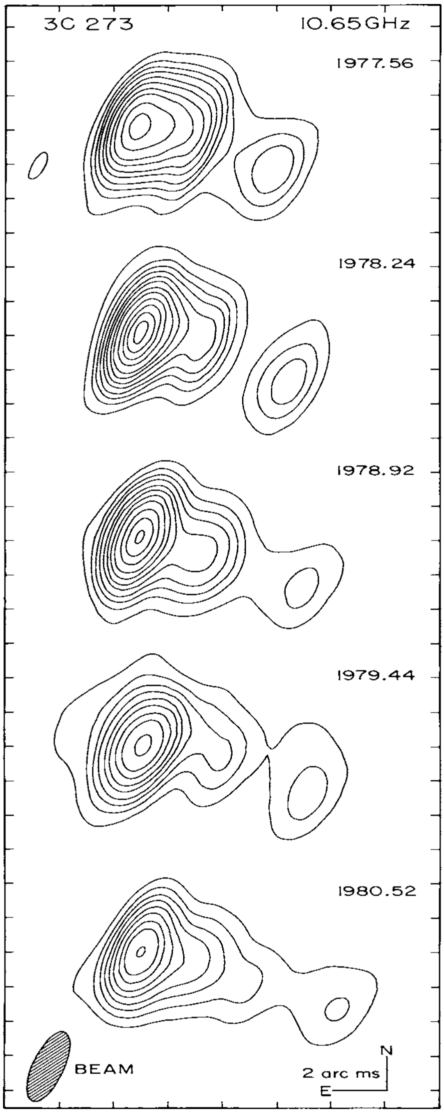

Figure 1.3 A Very Large Baseline Interferometry (VLBI) contour map of five epochs on an ejection event in the quasar 3C273 in the radio (10.65 GHz). (Reprinted by permission from the authors and *Nature*, Pearson, T. J. *et al.*, *Nature*, 280, 365. ©1981 Macmillan Publishers Ltd.)

where θ denotes the angle between the velocity of the blob and the line-of-sight. The projected distance on the celestial sphere is $D_\perp = \Delta t_e v \sin\theta$. The projected velocity on the sky is, therefore,

$$v_\perp = \frac{D_\perp}{\Delta t_r} = \frac{v \sin\theta}{1 - v\cos\theta}. \tag{1.29}$$

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Several limits can be deduced. The maximal value of the apparent velocity $v_\perp$ is

$$v_\perp = v\Gamma. \tag{1.30}$$

Thus, an observed value for $v_\perp$ gives a minimal value of the three-velocity and Lorentz factor

$$v = \frac{v_\perp}{\sqrt{1+v_\perp^2}}, \quad \Gamma = \sqrt{1+v_\perp^2}. \tag{1.31}$$

Similarly, an observed value for $v_\perp$ gives rise to a maximal angle θ upon setting $v = 1$. With (1.27), we conclude that the blob has a Lorentz factor $\Gamma \geq 10$.

1.5 Doppler shift

The combined effects of redshift and projection are known as Doppler shift. Consider harmonic wave-motion described by $e^{i\phi}$. The phase ϕ is a scalar, i.e. it is a Lorentz invariant. For a plane wave we have $\phi = k_a x^a = \eta_{ab} k^a x^b$ in terms of the wave four-vector k^a. Thus, k^a is a four-vector and transforms accordingly. A photon moving towards an observer with angle θ to the line-of-sight has $k^x = \epsilon \cos\theta$ for an energy $k^0 = \epsilon$. By the Lorentz transformation, the energy in the source frame with velocity v is given by

$$k'^0 = \Gamma(k^0 - vk^1), \tag{1.32}$$

so that

$$\epsilon' = \Gamma\epsilon(1 - v\cos\theta). \tag{1.33}$$

The result can be seen also by considering the arrival times of pulses emitted at the beginning and the end of a period of the wave. If T' and T denote the period, in the source and in the laboratory frame, respectively, then $2\pi = \epsilon' T' = \epsilon'(T/\Gamma)$. The two pulses have a difference in arrival times $\Delta t = T(1 - v\cos\theta)$ and the energy in the observer's frame becomes

$$\epsilon = \frac{2\pi}{\Delta t} = \frac{\epsilon'}{\Gamma(1 - v\cos\theta)}. \tag{1.34}$$

This is the same as (1.33).

1.6 Relativistic equations of motion

Special relativity implies that all physical laws obey the same local causal structure defined by light cones. This imposes the condition that the world-line of any particle through a point remains inside the local light cone. This is a geometrical

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description of the condition that all physical particles move with velocities less than (if massive) or equal to (if massless) the velocity of light.

Newton's laws of motion for a particle of mass m are given by the three equations

$$F_i = m\frac{d^2 x_i(t)}{dt^2} \quad (i = 1, 2, 3). \tag{1.35}$$

We conventionally use Latin indices from the middle of the alphabet to denote spatial components i, corresponding to (x, y, z). The velocity $dx_i(t)/dt$ is unbounded in response to a constant forcing (m is a constant), and we note that (1.35) consists of merely three equations motion. It follows that (1.35) does not satisfy causality, and is not Lorentz-invariant.

Minkowski's world-line x^b of a particle is generated by a tangent given by the velocity four-vector (1.9). Here, we use a normalization in which τ denotes the eigentime, (1.10). We consider the Lorentz-invariant equations of motion

$$f^b = \frac{dp^b}{d\tau}, \tag{1.36}$$

where

$$p^b = mu^b = (E, P^i) \tag{1.37}$$

denotes the particle's four-momentum in terms of its energy, conjugate to the time-coordinate t, and three-momentum, conjugate to the spatial coordinates x^i. There is one Lorentz invariant:

$$p^2 = -m^2, \tag{1.38}$$

which is an integral of motion of (1.36). The forcing in (1.36) is subject to the orthogonality condition $f^b p_b = 0$, describing orthogonality to its world-line.

The non-relativistic limit corresponding to small three-velocities v in (1.38) gives

$$E = \sqrt{m^2 + P^2} \simeq m + \frac{1}{2}mv^2. \tag{1.39}$$

We conclude that E represents the sum of the Newtonian kinetic energy and the mass of the particle. This indicates that m (i.e. mc^2) represents rest mass-energy of a particle. As demonstrated by nuclear reactions, rest mass-energy can be released in other forms of energy, and notably so in radiation. In general, it is important to note that energy is the time-component of a four-vector, and that it transforms accordingly.

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