Ricci's Theorem Proof :

$$B_{mn;p} \equiv \frac{\partial B_{mn}}{\partial x^p} - \Gamma^k_{mp} B_{kn} - \Gamma^k_{np} B_{mk}, \text{ 3rd - order covariant tensor, as before}$$

$$\Longrightarrow g_{mn;p} \equiv \frac{\partial g_{mn}}{\partial x^p} - \Gamma^k_{mp}\, g_{kn} - \Gamma^k_{np}\, g_{mk}, \text{ using Christoffel's of the 2nd - kind}$$

$$\equiv \frac{\partial g_{mn}}{\partial x^p} - \Gamma_{n,mp} - \Gamma_{m,np}, \text{ converted to Christoffel's of the 1st - kind}$$

$$\equiv \frac{\partial g_{mn}}{\partial x^p} - \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial x^p} + \frac{\partial g_{pn}}{\partial x^m} - \frac{\partial g_{mp}}{\partial x^n}\right) - \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial x^p} + \frac{\partial g_{pm}}{\partial x^n} - \frac{\partial g_{np}}{\partial x^m}\right)$$

$$\equiv \frac{\partial g_{mn}}{\partial x^p} - \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial x^p} + \frac{\partial g_{mn}}{\partial x^p} + \frac{\partial g_{pn}}{\partial x^m} - \frac{\partial g_{np}}{\partial x^m} + \frac{\partial g_{pm}}{\partial x^n} - \frac{\partial g_{mp}}{\partial x^n}\right)$$

$$\equiv \frac{\partial g_{mn}}{\partial x^p} - \frac{1}{2}\left(2\frac{\partial g_{mn}}{\partial x^p} + 0 + 0\right)$$

$$\equiv \frac{\partial g_{mn}}{\partial x^p} - \frac{\partial g_{mn}}{\partial x^p}$$

$$\Longrightarrow \therefore\ g_{mn;p} = 0, \text{ the covariant derivative of the metric tensor } g_{mn;p} \text{ vanishes!}$$

$$g^{mn} g_{kn} = \delta^m_k = \begin{cases} 0, & \text{if} \quad m \neq k \\ 1, & \text{if} \quad m = k \end{cases}, \text{ \underline{Conjugate Matrix} by Kronecker's delta}$$

$$\Longrightarrow (g^{mn} g_{kn})_{;p} = (\delta^m_k)_{;p} = 0, \text{ since covariant derivative of any constant is zero}$$

$$\Longrightarrow g^{mn}{}_{;p}\, g_{kn} + g^{mn} \overbrace{g_{kn;p}}^{\substack{0\\ \|}} = 0, \text{ distributing covariant derivative by Leibniz Rule}$$

$$\Longrightarrow g^{mn}{}_{;p}\, g_{kn} + g^{mn}(0) = 0, \text{ covariant derivative of covariant metric tensor is zero}$$

$$\Longrightarrow g^{mn}{}_{;p}\, g_{kn} = 0$$

$$\Longrightarrow g^{kr} g^{mn}{}_{;p}\, g_{kn} = 0, \text{ multiplying by dummy } g^{kr}$$

$$\Longrightarrow g^{mn}{}_{;p}\, g^{kr}\, g_{kn} = 0, \text{ rearranging terms}$$

$$\Longrightarrow g^{mn}{}_{;p}\, \delta^r_n = 0,\ \delta^r_n = \begin{cases} 0, & \text{if} \quad r \neq n \\ 1, & \text{if} \quad r = n \end{cases}$$

$$\Longrightarrow \therefore \boxed{g^{mn}{}_{;p} = 0}, \text{ covariant derivative of inverse metric tensor also vanishes!}$$

$$\Longrightarrow \therefore\ (g_{mn} A^n)_{;p} = g_{mn} A^n{}_{;p} = A_{m;p}, \text{ thus, \underline{all forms of the metric tensor behave like constants} under covariant derivation.}$$

Also, see: The Geodesic Spacetime Equation #10 - 'Christoffel Symbols ... again'

end